

MINUSCULE SCHUBERT VARIETIES AND MIRROR SYMMETRY

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ABSTRACT. We study the mirror symmetry for smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties using their degenerations to Hibi toric varieties. Listing all these Calabi–Yau 3-folds up to deformation equivalences, we find a new example of smooth Calabi–Yau 3-folds of Picard number one, which is a complete intersection in a locally factorial Schubert variety Σ of the Cayley plane $\mathbb{O}\mathbb{P}^2$. We calculate topological invariants and BPS numbers of this Calabi–Yau 3-fold and conjecture that it has a non-trivial Fourier–Mukai partner.

1. INTRODUCTION

Thousands examples of mirror pairs of Calabi–Yau 3-folds have given a driving power to the researches of mirror symmetry phenomenon. The earliest examples of mirror-constructed Calabi–Yau 3-folds include a quintic 3-fold [GP], which is appeared in relation to a conformal field theory in theoretical physics and remains a leading example of mirror symmetry, together with complete intersections in the product of projective spaces and weighted projective spaces. Further, we have a plenty examples as hypersurfaces [Bat1] and complete intersections [Bor] in Gorenstein toric Fano varieties. The mirror construction for these examples is based on the combinatorial duality of reflexive polytopes and nef-partitions. An extension of this mirror construction via conifold transition gives other examples as complete intersections in Grassmannians [BCFKvS1], partial flag manifolds [BCFKvS2] and smoothing of terminal hypersurfaces in Gorenstein toric Fano 4-folds [BK]. The mirror construction in these cases is also formulated by [Bat2] as a kind of monomial-divisor correspondence [AGM].

Studying this mirror construction via conifold transitions, we find a new example of smooth Calabi–Yau 3-fold in a Schubert variety of the Cayley plane $\mathbb{O}\mathbb{P}^2$, which seems to have a non-trivial Fourier–Mukai partner. More generally, we study complete intersection Calabi–Yau 3-folds in the so-called minuscule Schubert varieties in this paper (cf. Definition 2.2) and obtain the following results:

Main results (Proposition 3.1 and Proposition 4.1). *There is a unique deformation equivalent class of smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties, except for 11 known examples in homogeneous spaces. This Calabi–Yau 3-fold X is a general complete intersection of hyperplanes in a Schubert variety Σ in the Cayley plane $\mathbb{O}\mathbb{P}^2$ with $h^{1,1}(X) = 1$ and $h^{2,1}(X) = 52$, whose topological invariants are*

$$\deg(X) = 33, \quad c_2(X) \cdot H = 78, \quad \chi(X) = -102,$$

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where H is the ample generator of the Picard group $\text{Pic}(X) \cong \mathbb{Z}$.

The Schubert variety $\Sigma \subset \mathbb{OP}^2$ above is a 12 dimensional locally factorial Fano variety of Picard number one (cf. Definition 2.5 and Proposition 2.9). The Calabi–Yau 3-fold $X = \Sigma(1^9)$ in Main results, a general complete intersection of nine hyperplanes, is our main interest because of its mirror symmetric property. We study a family of affine complete intersections in $(\mathbb{C}^*)^n$ which are expected to be birational to the conjectural mirror manifolds of X [BCFKvS1] and obtain the Picard–Fuchs equation annihilating the fundamental period. The properties of the monodromy of this differential equation admit the geometric interpretation around another maximally unipotent monodromy (MUM) point $x = \infty$ and leads us to Conjecture 5.6, the existence of a non-trivial Fourier–Mukai partner. The integrality of the BPS numbers also supports this conjecture.

The paper is organized as follows.

In Section 2 we give some preliminaries. We deal with generalities on posets and distributive lattices, geometry of minuscule Schubert varieties and Hibi toric varieties, and toric degenerations. In particular, we bring in the combinatorial description of singular loci of minuscule Schubert varieties and Hibi toric varieties, which plays a key role for the proof of Main results.

In Section 3 we make a list of all the deformation equivalent (diffeomorphic) classes of smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties. We will see that there is a unique new example of such Calabi–Yau 3-folds, embedded in a locally factorial Schubert variety Σ in the Cayley plane \mathbb{OP}^2 (Proposition 3.1).

In Section 4 we give a computational method of calculating topological invariants for a kind of Calabi–Yau 3-folds of Picard number one which degenerate to complete intersection in Gorenstein Hibi toric varieties. We work on $\Sigma(1^9)$ as an example, and give a proof of Main results (Proposition 4.1). The topological Euler number is computed by using a conifold transition.

In Section 5 we study the mirror symmetry of complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties. We construct special one parameter families of affine complete intersections in $(\mathbb{C}^*)^n$ conjecturally birational to the mirror families (Conjecture 5.2) and give an expression for the fundamental periods (Proposition 5.4). We focus on the example $X = \Sigma(1^9)$ and obtain the Picard–Fuchs equation for its mirror, which suggests the existence of a non-trivial Fourier–Mukai partner of X (Conjecture 5.6). We also perform the monodromy calculation and the computation of BPS numbers using mirror symmetry. Every result seems very similar to that happened for the examples of the Pfaffian-Grassmannian [Rød] [HK] and the Reye congruence Calabi–Yau 3-fold [HT1, HT2].

In Appendix A, we treat other examples in minuscule homogeneous spaces.

In Appendix B, we put the table of BPS numbers calculated for $X = \Sigma(1^9)$ and its conjectural Fourier–Mukai partner X' .

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2. PRELIMINARIES

2.1. Posets and distributive lattices. A partially ordered set is called a *poset* for short. Let u and v be distinct elements of a poset P . There are three possibilities that $u \prec v$, $v \prec u$ or these elements are incomparable, written as $u \not\prec v$. We say that u *covers* v if $u \succ v$ and there is no $w \in P$ with $u \succ w \succ v$. The *Hasse diagram* of a poset P is the oriented graph with vertex set P , having an edge $e = \{u, v\}$ going down from u to v whenever u covers v in P . We can read many informations for a poset by drawing the Hasse diagram in a plane. For a poset P , an *order ideal* is a subset $I \subset P$ with the property that

$$u \in I \text{ and } v \prec u \text{ imply } v \in I.$$

A *lattice* L is a poset for which each pair of elements $\alpha, \beta \in L$ has the least upper bound $\alpha \vee \beta$ (called the join) and the greatest lower bound $\alpha \wedge \beta$ (called the meet) in L . A *distributive lattice* is a lattice on which the following identity holds for all triple elements $\alpha, \beta, \gamma \in L$

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

It is easy to see that a finite lattice L has the unique maximal and minimal element with respect to the partial order on L . In a lattice L , an element α is said to be *join irreducible* if α is neither the minimal element nor the join of a finite set of other elements.

For a finite poset P , the order ideals of P form a distributive lattice $J(P)$ with the partial order given by set inclusions. The join and the meet on $J(P)$ correspond to the set union and the set intersection, respectively. On the other hand, the full subposet of join irreducible elements of $J(P)$ coincides with P by the Birkhoff representation theorem [Bir]. In fact, this gives a one-to-one correspondence between finite posets and finite distributive lattices. We give an example of this correspondence in the following.

Example 2.1. We introduce the example which appears repeatedly throughout this paper. The Hasse diagrams of P and $J(P)$ are described in Figure 1. The distributive lattice $J(P)$ is obtained by drawing the Hasse diagrams for each order ideals of P explicitly. In particular, the maximal element and the minimal element of $J(P)$ correspond to the whole P and the empty set \emptyset , respectively. A circled vertex in $J(P)$ represents a join irreducible element of $J(P)$, i.e. the vertex with exactly one edge below. We can easily reconstruct the poset P as the set of circled vertices with the induced order in $J(P)$.

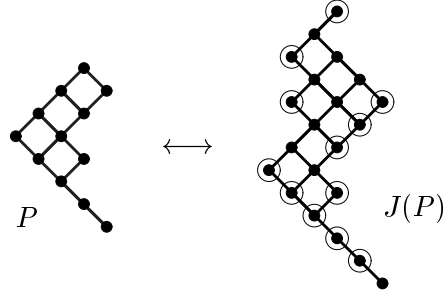


FIGURE 1. The Birkhoff correspondence

Now we introduce further definitions on a finite poset. Denote that the source $s(e) = u$ and the target $t(e) = v$ for an edge $e = \{u, v\}$ of the Hasse diagram of P if u covers v . A *chain* of length k in P is a sequence of elements $u_0 \prec u_1 \prec \cdots \prec u_k \in P$. A chain is called *maximal* if there is no $v \prec u_0$ or $w \succ u_k$ in P and u_i covers u_{i-1} for all $1 \leq i \leq k$. We call a finite poset P is *pure* if every maximal chain has the same length. Let us define the associated bounded poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ for any finite poset P , by extending the partial order on P with $\hat{0} \prec u \prec \hat{1}$. Every finite poset P has a height function h by defining $h(u)$ to be the length of the longest chain bounded above by u in \hat{P} . We also define the *height* h_P of P as $h(\hat{1})$. For example, $h_P = 9$ for the pure poset P in Example 2.1.

2.2. Minuscule Schubert varieties.

2.2.1. We give a brief summary for minuscule homogeneous spaces and minuscule Schubert varieties. A basic reference of this subject is [LMS]. Let G be a simply connected simple complex algebraic group, B a Borel subgroup and T a maximal torus in B . We denote by R^+ the set of positive roots and by $S = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. Let W be the Weyl group of G . Denote by Λ the character group of T , also called the weight lattice of G . The weight lattice Λ is generated by the fundamental weights $\lambda_1, \dots, \lambda_n$ defined by $(\alpha_i^\vee, \lambda_j) = \delta_{ij}$ for $1 \leq i, j \leq n$, where (\cdot, \cdot) is a W -invariant inner product and $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. An integral weight $\lambda = \sum n_i \lambda_i \in \Lambda$ is said to be dominant if $n_i \geq 0$ for all $i = 1, \dots, n$. For an integral dominant weight $\lambda \in \Lambda$, we denote by V_λ the irreducible G -module of the highest weight λ . The associated homogeneous space G/Q of λ is the G -orbit of the highest weight vector in the projective space $\mathbb{P}(V_\lambda)$, where $Q \supset B$ is the associated parabolic subgroup of G . A Schubert variety in G/Q is the closure of a B -orbit in G/Q . Recall the following definition of a minuscule weight, a minuscule homogeneous space and minuscule Schubert varieties.

Definition 2.2 (cf. [LMS, Definition 2.1]). Let $\lambda \in \Lambda$ be a fundamental weight. We call λ *minuscule* if it satisfies the following equivalent conditions.

- (1) Every weight of V_λ is in the orbit $W\lambda \subset \Lambda$.
- (2) $(\alpha^\vee, \lambda) \leq 1$ for all $\alpha \in R^+$.

The homogeneous space G/Q associated with a minuscule weight λ is said to be minuscule. The Schubert varieties in minuscule G/Q are also called minuscule.

Let us recall that a parabolic subgroup $Q \supset B$ is determined by a subset S_Q of S associated with negative root subgroups. A useful notation for a homogeneous space G/Q is to cross the nodes in the Dynkin diagram which correspond to the simple roots in $S \setminus S_Q$. With this notation, the minuscule homogeneous spaces are as shown in Table 1. This contains the Grassmannians $G(k, n)$, the orthogonal Grassmannians $OG(n, 2n)$, even dimensional quadrics Q^{2n} and, finally, the Cayley plane $\mathbb{OP}^2 = E_6/Q_1$ and the Freudenthal variety E_7/Q_7 , where we use the Bourbaki labelling for the roots. We omit two kind of minuscule weights for groups of type B and type C , since they give the isomorphic varieties to those for simply laced groups.

| $G(k, n+1)$ | $OG(n, 2n)$ | Q^{2n-2} | \mathbb{OP}^2 | E_7/Q_7 |
|-------------|-------------|------------|-----------------|-----------|
| | | | | |

TABLE 1. Minuscule homogeneous spaces

The Weyl group W is generated by simple reflections $s_\alpha \in W$ for $\alpha \in S$. These generators define the length function l on W . Let us denote by W_Q the Weyl group of Q , i.e. the subgroup generated by $\{s_\alpha \in W \mid \alpha \in S_Q\}$, and by W^Q the set of minimal length representatives of the coset W/W_Q in W . For any $w \in W^Q$, we denote by $X(w) = \overline{BwQ/Q}$ the Schubert variety in G/Q associated with w , which is a $l(w)$ -dimensional normal Cohen–Macaulay projective variety with at worst rational singularities. There is a natural partial order \prec on W^Q called the Bruhat order, defined as $w_1 \preceq w_2 \Leftrightarrow X(w_1) \subset X(w_2)$. We recall the following fundamental fact for minuscule homogeneous spaces.

Proposition 2.3 ([Pro, Proposition V.2]). *For a minuscule homogeneous space G/Q , the poset W^Q is a finite distributive lattice.*

From Proposition 2.3 and the Birkhoff representation theorem, we can define the *minuscule poset* P_Q for the minuscule G/Q such that $J(P_Q) = W^Q$ as in [Pro]. Moreover, the order ideal $P_w \subset P_Q$ associated with $w \in W^Q$ is called the minuscule poset for the minuscule Schubert variety $X(w) \subset G/Q$. For example, the order ideals $P_Q, \emptyset \subset P_Q$ turn out to be the minuscule posets for the total space $X(w_Q) = G/Q$ and the B -fixed point $X(\text{id}) = Q/Q$, respectively, where w_Q is the unique longest element in W^Q . The minuscule poset for minuscule Schubert varieties is a generalization of the Young diagram for Grassmann Schubert varieties.

Example 2.4. An easy method to compute the Hasse diagram of W^Q is to trace out the W -orbit of certain dominant weight whose stabilizer coincides with W_Q (cf. [BE, §4.3]). Denote by $(ij \cdots k)$ the element $w = s_{\alpha_i} s_{\alpha_j} \cdots s_{\alpha_k} \in W$, where s_α is simple reflection with respect to $\alpha \in S$. The initial part of the Hasse diagram of W^Q for the Cayley plane \mathbb{OP}^2 is the following

$$\begin{array}{c} \text{id} - (1) - (31) - (431) \begin{array}{l} \swarrow (5431) \searrow (2431) \end{array} \begin{array}{l} \swarrow (65431) \searrow (25431) \end{array} \begin{array}{l} \swarrow (265431) \searrow (425431) \end{array} \begin{array}{l} \swarrow (4265431) \searrow (3425431) \end{array} \cdots \\ \cdots \end{array}$$

where the right covers the left for connected two elements with respect to the Bruhat order. Thus we obtain the Hasse diagram of the distributive lattice W^Q and hence the minuscule poset for every Schubert variety in \mathbb{P}^2 .

Definition 2.5. In the above notation, let us set $w = (345134265431) \in W^Q$. We denote by Σ the associated 12-dimensional Schubert variety $X(w)$ in the Cayley plane \mathbb{P}^2 , which corresponds to the minuscule poset P in Example 2.1.

Remark 2.6. We remark on another geometric characterization of minuscule homogeneous spaces in [LMS, Definition 2.1]. A fundamental weight λ is minuscule (Definition 2.2) if and only if the following condition holds.

(3) For the associated homogeneous space G/Q , the Chevalley formula

$$[H] \cdot [X(w)] = \sum_{w \text{ covers } w'} [X(w')]$$

holds for all $w \in W^Q$ in the Chow ring of G/Q , where H is the unique Schubert divisor in G/Q and W^Q is the poset with the Bruhat order.

As a corollary, it turns out that the degree of a minuscule Schubert variety $X(w)$ with respect to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ equals the number of maximal chains in $J(P_w)$. For example, we obtain $\deg \Sigma = 33$ by counting the maximal chains in $J(P)$ in Figure 1.

2.2.2. Next we introduce further definitions for the description of singularities of minuscule Schubert varieties. As we expect from the computation in Example 2.4, the Bruhat order on W^Q is generated by simple reflections for minuscule G/Q [LW, Lemma 1.14], that is,

$$w_1 \text{ covers } w_2 \Leftrightarrow w_1 = s_\alpha \cdot w_2 \text{ and } l(w_1) = l(w_2) + 1 \text{ for some } \alpha \in S.$$

From this fact, a join irreducible element $u \in W^Q$ covers the unique element $s_{\beta_Q(u)} \cdot u \in W^Q$ where $\beta_Q(u) \in S$. Thus we can define the natural coloration $\beta_Q : P_Q \rightarrow S$ for a minuscule poset P_Q by simple roots S . We also define the coloration β_w on each minuscule poset $P_w \subset P_Q$ by restricting β_Q on P_w . The minuscule poset P_w with the coloration $\beta_w : P_w \rightarrow S$ has in fact the same information as the minuscule quiver introduced by Perrin [Per1, Per2], which gives a good description of geometric properties of minuscule Schubert varieties $X(w)$. Now we translate the combinatorial notions and useful facts on the geometry of minuscule Schubert varieties $X(w)$ from [Per1, Per2] in our terminology.

Definition 2.7. Let P be a minuscule poset with the coloration $\beta : P \rightarrow S$.

- (1) A *peak* of P is a maximal element u in P .
- (2) A *hole* of P is a maximal element u in $\beta^{-1}(\alpha)$ for some $\alpha \in S$ such that there are exactly two elements $v_1, v_2 \in P$ with $u \prec v_i$ and $(\beta(u)^\vee, \beta(v_i)) \neq 0$ ($i = 1, 2$).

Let us denote by $\text{Peaks}(P)$ and $\text{Holes}(P)$ the set of peaks and holes of P , respectively. A hole u of the poset P is said to be *essential* if the order ideal $P^u := \{v \in P \mid v \not\prec u\}$ contains all other holes in P .

Let $X(w)$ be a minuscule Schubert variety in G/Q and P_w the associated minuscule poset. Weil and Cartier divisors on $X(w)$ are described in terms of the poset P_w . In fact, it is clear

that any Schubert divisor coincides with a Schubert variety D_u associated with P_w^u for some $u \in \text{Peaks}(P_w)$. It is well-known that the divisor class group $\text{Cl}(X(w))$ is the free \mathbb{Z} -module generated by the classes of the Schubert divisors D_u for $u \in \text{Peaks}(P_w)$, and the Picard group $\text{Pic}(X(w))$ is isomorphic to \mathbb{Z} generated by $\mathcal{O}_{G/Q}(1)|_{X(w)}$. As we saw in Remark 2.6, the Cartier divisor corresponding to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ is

$$\sum_{u \in \text{Peaks}(P_w)} D_u.$$

We use the following results by Perrin about the singularities of $X(w)$.

Proposition 2.8 ([Per1, Per2]). *Let $X(w)$ be a minuscule Schubert variety and P_w the associated minuscule poset.*

- (1) [Per1, Proposition 4.17] *An anticanonical Weil divisor of $X(w)$ is*

$$-K_{X(w)} = \sum_{u \in \text{Peaks}(P_w)} (h(u) + 1) D_u.$$

In particular, $X(w)$ is Gorenstein if and only if P_w is pure. In this case $X(w)$ is a Fano variety of index h_{P_w} .

- (2) [Per2, Theorem 2.7 (1)] *The Schubert subvariety associated with the order ideal $P_w^u \subset P_w$ for an essential hole u of P_w is an irreducible component of the singular loci of $X(w)$. All the irreducible components of the singular loci are obtained in this way.*

We apply Proposition 2.8 to our main example Σ and obtain the following.

Proposition 2.9. *Let Σ be the minuscule Schubert variety in \mathbb{OP}^2 defined by Definition 2.5.*

- (1) *Σ is a locally factorial Gorenstein Fano variety of index 9.*
(2) *The singular locus of Σ is isomorphic to \mathbb{P}^5 .*

Proof. The former holds because the corresponding minuscule poset P (Figure 1) is pure with $h_P = 9$ and the unique peak. From the computation of the Hasse diagram of W^Q of \mathbb{OP}^2 in Example 2.4, the coloration $\beta : P \rightarrow S$ is given as the following picture.

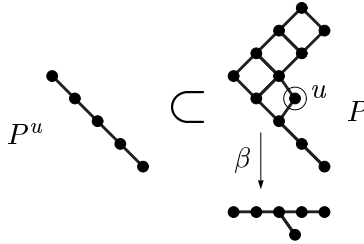


FIGURE 2. The coloration of the minuscule poset P and the singular locus

A unique (essential) hole of P is the circled vertex u , whose color is $\alpha_2 \in S$. The corresponding Schubert subvariety is described by the minuscule poset P^u , which coincides with the singular locus of Σ by Proposition 2.8. It is isomorphic to \mathbb{P}^5 because the degree equals to one. \square

Finally, we record the useful vanishing theorems for minuscule Schubert varieties.

Theorem 2.10 ([LMS, Theorem 7.1]). *Let λ be a minuscule weight, $G/Q \subset \mathbb{P}(V_\lambda)$ the associated homogeneous space and $X(w) \subset G/Q$ a minuscule Schubert variety.*

- (1) $H^0(\mathbb{P}(V_\lambda), \mathcal{O}(m)) \rightarrow H^0(X(w), \mathcal{O}(m))$ is surjective for all $m \geq 0$,
- (2) $H^i(X(w), \mathcal{O}(m)) = 0$ for all $m \in \mathbb{Z}$ and $0 < i < l(w)$,
- (3) $H^{l(w)}(X(w), \mathcal{O}(m)) = 0$ for all $m \geq 0$.

2.3. Hibi toric varieties.

2.3.1. We introduce another kind of projective varieties, called Hibi toric varieties, deeply related with the minuscule Schubert varieties.

First let us define the polytope $\Delta(P)$ called the *order polytope* for a finite poset P . These polytopes have been studied in a number of literature, for example [Sta]. Let $N = \mathbb{Z}P$ and $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the free \mathbb{Z} -modules of rank $|P|$, and $M_{\mathbb{R}}$ the real scalar extension of M . The order polytope $\Delta(P)$ is defined in $M_{\mathbb{R}}$ as follows.

$$\Delta(P) := \left\{ x = \sum_{u \in P} x_u \chi_u \in [0, 1]^P \mid x_u \leq x_v \text{ for all } u \prec v \in P \right\},$$

where $[0, 1]^P \subset M_{\mathbb{R}}$ is a $|P|$ -dimensional unit cube of all real maps $x : P \rightarrow [0, 1]$, and $\{\chi_u \mid u \in P\} \subset M$ is the dual basis of $P \subset N$, i.e. $\chi_u(v) = \delta_{uv}$ for $u, v \in P$. It is easy to see that the order polytope $\Delta(P)$ is an integral convex polytope of dimension $|P|$.

Definition 2.11. We call the projective toric variety $\mathbb{P}_{\Delta(P)}$ defined by an order polytope $\Delta(P) \subset M_{\mathbb{R}}$ the *(projective) Hibi toric variety* for the poset P .

Example 2.12. One of the simplest examples of order polytopes is the so-called Gelfand–Tsetlin polytopes for fundamental weights of special linear groups $SL(n+1, \mathbb{C})$. In general, the Gelfand–Tsetlin polytope for an integral dominant weight $\lambda = (\Lambda_0, \dots, \Lambda_n) \in \mathbb{Z}^{n+1} / \langle (1, \dots, 1) \rangle \simeq \Lambda$ is defined by the following inequalities in $\mathbb{R}^{n(n+1)/2}$

$$\Lambda_n \leq x_{i+1, j+1} \leq x_{i, j} \leq x_{i, j+1} \leq \Lambda_i \quad \text{for all } 0 \leq i \leq j \leq n-1.$$

These inequalities can be represented in a diagram, like as the left of Figure 3 for $n = 3$.

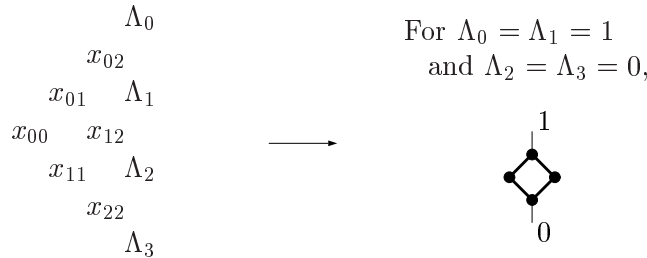


FIGURE 3. the Gelfand–Tsetlin polytopes for $SL(4, \mathbb{C})$

In fact, the Gelfand–Tsetlin polytope for a fundamental weight $\lambda = (1, \dots, 1, 0, \dots, 0)$ is the order polytope for a poset of rectangle shape, like as the right of Figure 3 for example.

It coincides with the minuscule poset for a Grassmannian of type A . The corresponding Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is the toric variety $P(k, n+1)$ defined by [BCFKvS1].

Remark 2.13. There is the alternative definition of the Hibi toric variety using its homogeneous coordinate ring, which is standard in the literature. Let $J(P)$ be the distributive lattice of order ideals of a finite poset P . Denote by $\mathbb{C}[J(P)]$ the polynomial ring over \mathbb{C} in $|J(P)|$ indeterminates p_α ($\alpha \in J(P)$). Let $I(J(P)) \subset \mathbb{C}[J(P)]$ the homogeneous ideal generated by the following binomial relations:

$$p_\tau p_\phi - p_{\tau \wedge \phi} p_{\tau \vee \phi} \quad (\tau \not\prec \phi).$$

One can check that the graded algebra $A_{J(P)} := \mathbb{C}[J(P)]/I(J(P))$ with the standard \mathbb{Z} -grading inherited from $\mathbb{C}[J(P)]$ coincides with the homogeneous coordinate ring of the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ with the embedding defined by the very ample line bundle associated with $\Delta(P)$. The graded algebra $A_{J(P)}$ is usually called the *Hibi algebra* on the distributive lattice $J(P)$ (cf. [Hib]).

2.3.2. A nice property of Hibi toric varieties is that torus invariant subvarieties in Hibi toric varieties are also Hibi toric varieties. We need an explicit description of these subvarieties and their singularities. The subject was well studied in [Wag] and also in [BL1, BL2] and [LM].

Let us recall the inequalities in the definition of an order polytope $\Delta(P)$. They are generated by $x_{s(e)} \geq x_{t(e)}$ for all $e \in E$, where E is the set of edges of the Hasse diagram of $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ and $x_{\hat{0}} = 0$ and $x_{\hat{1}} = 1$. We can get a face of $\Delta(P)$ by replacing some of these inequalities with equalities. To describe such a replacement, we introduce some further concepts for posets.

Recall that a *full subposet* $y \subset \hat{P}$ is a subset of \hat{P} whose poset structure is that inherited from \hat{P} . We call a full subposet $y \subset \hat{P}$ *connected* if all the elements in y are connected by edges in the Hasse diagram of y , and *convex* if $u, v \in y$ and $u \prec w \prec v$ imply $w \in y$.

Definition 2.14. A surjective map $f : \hat{P} \rightarrow \hat{P}'$ is called a *contraction* of the bounded poset \hat{P} if every fiber $f^{-1}(i)$ ($i \in \hat{P}'$) is connected full subposet of \hat{P} not containing both $\hat{0}$ and $\hat{1}$, and the following condition holds for all $u_k, v_k \in f^{-1}(k)$ and $i \neq j \in \hat{P}'$:

$$\text{a relation } u_i \prec u_j \text{ implies } v_i \not\prec v_j.$$

Remark 2.15. A contraction $f : \hat{P} \rightarrow \hat{P}'$ gives a natural partial order on the image set \hat{P}' , i.e. the partial order generated by the following relations:

$$i \prec j \Leftrightarrow \text{there exist } u \in f^{-1}(i) \text{ and } v \in f^{-1}(j) \text{ such that } u \prec v \text{ in } \hat{P}.$$

Further, \hat{P}' turns out to be a bounded poset by setting $\hat{1} \in f^{-1}(\hat{1})$ and $\hat{0} \in f^{-1}(\hat{0})$. Hence in fact, the above definition of contraction coincides with the more abstract definition in [Wag]; the fiber-connected tight surjective morphism of bounded posets.

For a contraction $f : \hat{P} \rightarrow \hat{P}'$, the corresponding face of $\Delta(P)$ is given by

$$\theta_f := \left\{ x \in \Delta(P) \mid x_u = x_v \text{ for all } u, v \in f^{-1}(i) \text{ and } i \in \hat{P}' \right\}.$$

Conversely, we can reconstruct the contraction from each face $\theta_f \subset \Delta(P)$ by looking at the coordinates of general point in θ_f . Now we can rephrase the classical fact on the face structure of order polytopes in our terminology (cf. [Wag, Theorem 1.2]).

Proposition 2.16. *Let P be a finite poset, and $\Delta(P)$ the associated order polytope. The above construction gives a one-to-one correspondence between the faces of $\Delta(P)$ and the contractions of \hat{P} . Moreover, an inclusion of the faces corresponds to a composition of contractions.*

Remark 2.17. It is obvious that the face $\theta_{\hat{P} \rightarrow \hat{P}'}$ $\subset \Delta(P)$ coincides with the $|P'|$ -dimensional order polytope $\Delta(P')$ under the suitable choice of subspace of $M_{\mathbb{R}}$ and the unimodular transformation. This means that the torus invariant subvarieties in Hibi toric varieties are also Hibi toric varieties as noted before.

Similarly to the case of minuscule Schubert varieties, Weil and Cartier divisors on a Hibi toric variety $\mathbb{P}_{\Delta(P)}$ are naturally described in terms of the poset P . We can show that the divisor class group $\text{Cl}(\mathbb{P}_{\Delta(P)})$ is the free \mathbb{Z} -module of rank $|E| - |P|$ and the Picard group $\text{Pic}(\mathbb{P}_{\Delta(P)})$ is the free \mathbb{Z} -module whose rank coincides with the number of connected components of the Hasse diagram of P . We use the following results on singularities of the Hibi toric varieties.

Proposition 2.18 ([HH, Remark 1.6 and Lemma 1.4]). *Let P be a finite poset. The Hibi algebra $A_{J(P)}$ is Gorenstein if and only if P is pure. In this case the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ is a Gorenstein Fano variety with at worst terminal singularities.*

Theorem 2.19 ([Wag, Theorem 2.3 and Proof of Corollary 2.4]). *Let $\mathbb{P}_{\Delta(P)}$ be a Hibi toric variety for a finite poset P . A face $\theta_f \subset \Delta(P)$ corresponds to an irreducible component of the singular loci of $\mathbb{P}_{\Delta(P)}$ if and only if one of the fiber $f^{-1}(i)$ of the contraction $f : \hat{P} \rightarrow \hat{P}'$ is a minimal convex cycle and all other fibers $f^{-1}(j)$ ($j \neq i$) consist of one element.*

Proposition 2.20 ([HW, Corollary of Lemma 5]). *Let P be a finite poset. The Poincaré series of the Hibi algebra $A_{J(P)}$ is written as the following form:*

$$\mathcal{P}(t) = \sum_{i=0}^{|P|} c_i (t/(1-t))^i,$$

where $c_0 = 1$ and c_i ($i \neq 0$) is the number of the chains of length i in $J(P)$. In particular, the degree of $\mathbb{P}_{\Delta(P)} \subset \text{Proj } \mathbb{C}[J(P)]$ equals the number of maximal chains in $J(P)$.

2.4. Toric degenerations. Now we state the key theorem by Gonciulea and Lakshmibai in our terminology, which gives the relationship between minuscule Schubert varieties and Hibi toric varieties.

Theorem 2.21 ([GL, Theorem 7.34]). *A minuscule Schubert variety $X(w)$ degenerates to the Hibi toric variety $\mathbb{P}_{\Delta(P_w)}$, where P_w is the minuscule poset for $X(w)$. More precisely, there exists a flat family $\mathcal{V} \rightarrow \mathbb{C}$ such that $\mathcal{V}_t \simeq X(w)$ for all $t \in \mathbb{C}^*$ and $\mathcal{V}_0 \simeq \mathbb{P}_{\Delta(P_w)}$.*

Corollary 2.22. *A Gorenstein minuscule Schubert variety has at worst terminal singularities.*

Proof. By [HH] a Gorenstein Hibi toric variety has at worst terminal (and hence canonical) singularities, Proposition 2.18 (1). It is known that a deformation of a terminal singularity

is again terminal [Nak] as with the case of canonical singularities [Kaw]. Hence the assertion follows from Theorem 2.21. \square

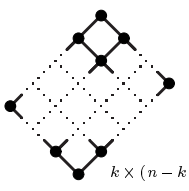
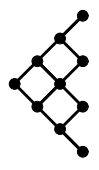
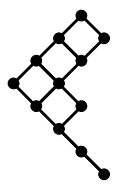
3. LIST OF COMPLETE INTERSECTION CALABI–YAU 3-FOLDS

In this section, we study the smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties. We show that there is a unique new deformation equivalent class of such Calabi–Yau 3-folds, that is, the complete intersection of nine hyperplanes in a locally factorial Schubert variety Σ defined by Definition 2.5.

First we fix some basic terminologies to clarify the meaning of our result. Let $X(w)$ be a minuscule Schubert variety. We call a subvariety $X \subset X(w)$ a *complete intersection* if it is the common zero locus of $r = \text{codim} X$ global sections of invertible sheaves on $X(w)$. We can denote by $X = X(w)(d_1, \dots, d_r)$ the complete intersection variety of general r sections of degree d_1, \dots, d_r with respect to $\mathcal{O}_{G/Q}(1)|_{X(w)}$ since $\text{Pic } X(w) \simeq \mathbb{Z}$. A *Calabi–Yau* variety X is a normal projective variety with at worst Gorenstein canonical singularities and with trivial canonical bundle $K_X \simeq 0$ such that $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. Two smooth varieties X_1 and X_2 are called *deformation equivalent* if there exist a smooth family $\mathcal{X} \rightarrow U$ over a connected open base $U \subset \mathbb{C}$ such that $\mathcal{X}_{t_1} \simeq X_1$ and $\mathcal{X}_{t_2} \simeq X_2$ for some $t_1, t_2 \in U$. In this case, X_1 and X_2 turn out to be diffeomorphic.

Now we summarize all possible smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties:

Proposition 3.1. *A smooth complete intersection Calabi–Yau 3-fold in a minuscule Schubert variety is one of that listed in the following table up to deformation equivalence.*

| | | | |
|-------------------|---|---|---|
| minuscule posets |  |  |  |
| ambient varieties | $G(k, n)$ | $OG(5, 10)$ | Σ |
| degrees | 10 examples | $(1^6, 2)$ | (1^9) |

In this table, 10 known examples in Grassmannians of type A include five in projective spaces;

$$\mathbb{P}^4(5), \mathbb{P}^5(2, 4), \mathbb{P}^5(3^2), \mathbb{P}^6(2^2, 3) \text{ and } \mathbb{P}^7(2^4),$$

and five in others, whose mirror symmetry was discussed in [BCFKvS1];

$$G(2, 5)(1^2, 3), G(2, 5)(1, 2^2), G(2, 6)(1^4, 2), G(3, 6)(1^6) \text{ and } G(2, 7)(1^7).$$

For all these Calabi–Yau 3-folds, the Picard number equals to one.

Before the proof of Proposition 3.1, we show a lemma on the Hibi toric varieties to handle the complexity of the redundant appearance of deformation equivalent varieties. The special case of this lemma was already used in [BCFKvS1] to prove the mirror duality of Hodge numbers for a Calabi–Yau 3-fold $G(2,5)(1^2,3)$ and its mirror by reducing the argument to that of a hypersurface in a Hibi toric variety.

Lemma 3.2. *Let P be a finite poset and $P^* = P \cup \{\hat{1}\}$, $P_* = P \cup \{\hat{0}\}$ the posets where $u \prec \hat{1}$, $\hat{0} \prec u$ for all $u \in P$, respectively. The Hibi toric varieties $\mathbb{P}_{\Delta(P^*)}$ and $\mathbb{P}_{\Delta(P_*)}$ are the projective cones over $\mathbb{P}_{\Delta(P)}$ in $\text{Proj } \mathbb{C}[J(P^*)]$ and $\text{Proj } \mathbb{C}[J(P_*)]$, respectively.*

Proof. Let $1 = 1_{J(P^*)}$ be the maximal element in $J(P^*)$. This 1 is join irreducible in $J(P^*)$ because $J(P)$ has the unique maximal element. Therefore the variable p_1 in the homogeneous coordinate ring $A_{J(P^*)}$ of $\mathbb{P}_{\Delta(P^*)}$ does not involve in any relation $p_\tau p_\phi - p_{\tau \wedge \phi} p_{\tau \vee \phi}$ ($\tau \not\prec \phi$). Thus the Hibi toric variety $\mathbb{P}_{\Delta(P^*)}$ is the projective cone over $\mathbb{P}_{\Delta(P)}$. The same argument is also valid for the minimal element $0 = 0_{J(P_*)}$ in $J(P_*)$ by considering the order dual of P . \square

Proof of Proposition 3.1. We may assume that the ambient minuscule Schubert variety is Gorenstein. In fact, from the adjunction formula and the Grothendieck–Lefschetz theorem for divisor class groups of normal projective varieties [RS], we have an explicit formula of the canonical divisor as a Cartier divisor, $K_{X(w)} = -D_1 - \cdots - D_r$ where $D_j \subset X(w)$ is a very ample Cartier divisor of degree d_j and $X(w)(d_1, \dots, d_r)$ is a general Calabi–Yau complete intersection.

Let $X(w)$, $X(w')$ be minuscule Schubert varieties and P_w , $P_{w'}$ the corresponding minuscule posets, respectively. Assume that $P_{w'}$ coincides with a d -times iterated extension $(P_w)^{*\dots*}$ of P_w . From Lemma 3.2, it holds that $\mathbb{P}_{\Delta(P_w)}$ is isomorphic to a complete intersection of d general hyperplanes in $\mathbb{P}_{\Delta(P_{w'})}$. By Theorem 2.21, there exist the toric degenerations of $X(w)$ and $X(w')$ to the Hibi toric varieties $\mathbb{P}_{\Delta(P_w)}$ and $\mathbb{P}_{\Delta(P_{w'})}$, respectively. This means that general complete intersection Calabi–Yau 3-folds $X = X(w)(d_1, \dots, d_r)$ and $X' = X(w')(1^d, d_1, \dots, d_r)$ can be connected by flat deformations through a complete intersection $X_0 = \mathbb{P}_{\Delta(P_w)}(d_1, \dots, d_r)$. Since X_0 has at worst terminal singularities, the Kuranishi space is smooth by [Nam, Theorem A] and the degenerating loci have a positive complex codimension. Therefore X and X' are connected by smooth deformation. Thus we can eliminate such redundancy arisen from iterated extensions of minuscule posets.

A Gorenstein minuscule Schubert variety $X(w)$ with minuscule poset $P = P_w$ is a $|P|$ -dimensional Fano variety of index h_P as we stated in Proposition 2.8 (1). The condition for general complete intersections in $X(w)$ to be Calabi–Yau 3-folds gives a strong combinatorial restriction for the poset P as follows,

$$h_P - 1 \leq |P| \leq h_P + 3.$$

On the other hand, there is a complete list of the minuscule posets in [Per1]. Hence we can make a list of the complete intersection Calabi–Yau 3-folds by counting such posets.

We can check that the resulting 3-folds $X \subset X(w)$ with trivial canonical bundles turn out to be Calabi–Yau varieties after some computation using the vanishing theorems for $X(w)$, Theorem 2.10. We verify the smoothness of these 3-folds by looking at the codimension of

the singular loci of $X(w)$ using Proposition 2.8 (2). For example, a general linear section $X = \Sigma(1^9)$ is smooth since the singular loci of Σ have codimension 7 as we saw in Proposition 2.9. All the smooth cases are contained in locally factorial minuscule Schubert varieties, i.e. the minuscule poset P has the unique peak. Thus the Picard number equals to one again by the Grothendieck–Lefschetz theorem for divisor class groups [RS]. This completes the proof. \square

4. TOPOLOGICAL INVARIANTS

Now we explain our calculation of topological invariants for smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties by taking $X = \Sigma(1^9)$ as an example. The topological invariants mean the degree $\deg(X) = \int_X H^3$, the linear form associated with the second Chern class $c_2(X) \cdot H = \int_X c_2(X) \cup H$ and the Euler number $\chi(X) = \int_X c_3(X)$, where H is the ample generator of $\text{Pic}(X) \simeq \mathbb{Z}$. These three invariants characterize the diffeomorphic class of smooth simply connected Calabi–Yau 3-folds of Picard number one [Wal].

Proposition 4.1. *The topological invariants of $X = \Sigma(1^9)$ are*

$$\deg(X) = 33, \quad c_2(X) \cdot H = 78, \quad \chi(X) = -102.$$

Proof. The degree of X coincides with that of the minuscule Schubert variety $\Sigma \subset \mathbb{O}\mathbb{P}^2$ since the ample generator $\mathcal{O}_\Sigma(1)$ of $\text{Pic } \Sigma$ is the restriction of $\mathcal{O}_{\mathbb{O}\mathbb{P}^2}(1)$ and X is a linear section. We obtain $\deg(\Sigma) = 33$ by using the Chevalley formula of $\mathbb{O}\mathbb{P}^2$ as we already saw in Remark 2.6.

The Schubert variety $V^0 := \Sigma$ and its general complete intersections $V^j := \Sigma(1^j)$ have at worst rational singularities (in fact at worst terminal singularities by Corollary 2.22). Hence the Kawamata–Viehweg vanishing theorem gives

$$H^i(V^j, \omega_{V^j} \otimes \mathcal{O}_{V^j}(k)) = H^i(V^j, \mathcal{O}_{V^j}(k + j - 9)) = 0 \quad \text{for all } i > 0 \text{ and } k > 0.$$

Together with the long cohomology exact sequences of

$$0 \rightarrow \mathcal{O}_{V^j}(k) \rightarrow \mathcal{O}_{V^j}(k+1) \rightarrow \mathcal{O}_{V^{j+1}}(k+1) \rightarrow 0,$$

the holomorphic Euler number of $X = V^9$ becomes

$$\chi(X, \mathcal{O}_X(1)) = \dim H^0(X, \mathcal{O}_X(1)) = \dim H^0(\Sigma, \mathcal{O}_\Sigma(1)) - 9 = |J(P)| - 9 = 12.$$

On the other hand, it holds that

$$\chi(X, \mathcal{O}_X(1)) = \frac{1}{6} \deg(X) + \frac{1}{12} c_2(X) \cdot H$$

from the Hirzebruch–Riemann–Roch theorem of the smooth Calabi–Yau 3-fold X . Thus we obtain $c_2(X) \cdot H = 78$.

For the topological Euler number $\chi(X)$, we use the toric degeneration of Σ to the Hibi toric variety $\mathbb{P}_{\Delta(P)}$, Theorem 2.21. A general complete intersection Calabi–Yau 3-fold X_0 in the degenerated variety $\mathbb{P}_{\Delta(P)}$ has finitely many nodes by a Bertini type theorem for toroidal singularities. In fact, we know that three dimensional Gorenstein terminal toric singularities are at worst nodes. Thus we obtain a conifold transition Y of X , which is a smooth Calabi–Yau

complete intersection in a MPCP resolution $\widehat{\mathbb{P}}_{\Delta(P)}$ of $\mathbb{P}_{\Delta(P)}$ defined by [Bat1]. By Theorem 5.1, the Hodge numbers of Y can be calculated as $h^{1,1}(Y) = 5$ and

$$\begin{aligned} h^{2,1}(Y) &= 9(|J(P)| - 9) - \sum_{e \in E} (l^*(9\theta_e) - 9l^*(8\theta_e)) - |P| \\ &= 96 - \sum_{e \in E} (l^*(9\theta_e) - 9l^*(8\theta_e)). \end{aligned}$$

To count the number of interior integral points in each facet, we use Proposition 2.16 which states a face of the order polytope is also the order polytope of some poset P' . For each facet θ_e , the corresponding poset P' (or \hat{P}') is easily obtained by replacing the inequality $x_{s(e)} \geq x_{t(e)}$ by the equality $x_{s(e)} = x_{t(e)}$ and by considering the induced partial order. The Hasse diagram of resulting posets P' are shown in the following table, where the numbering of edges is chosen from the upper left in a picture of the Hasse diagram of \hat{P} like as in Figure 1.

| facets | θ_1 | θ_2 | θ_3, θ_6 | θ_4 | θ_5 | θ_7, θ_{10} | θ_8 | θ_9 | θ_{11} | θ_{12}, θ_{14} | θ_{13} | $\theta_{15}, \theta_{16}, \theta_{17}$ |
|------------------|------------|------------|----------------------|------------|------------|-------------------------|------------|------------|---------------|----------------------------|---------------|---|
| | | | | | | | | | | | | |
| $l^*(8\theta_i)$ | 1 | - | - | - | - | - | - | - | - | - | - | 1 |
| $l^*(9\theta_i)$ | 20 | 3 | 1 | 2 | - | 1 | - | 2 | - | 1 | 2 | 20 |

As shown in this table, some posets P' are pure and others are not pure. For a pure poset P' , the face $\theta_e \simeq \Delta(P')$ defines Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P')}$. Then we know $l^*(h_{P'}\theta_e) = 1$ and $l^*((h_{P'} + 1)\theta_e) = |J(P')|$. When P' is not pure, we can also easily obtain the number $l^*(k\theta_e)$ by counting the points satisfying the inequalities of the polytope $k\theta_e \simeq k\Delta(P')$ strictly. For example, $9\theta_2$ contains three internal integral points corresponding to

$$\begin{array}{c} 9 \\ x_1 \\ x_2 \\ x_3 \ x_4 \\ x_5 \ x_6 \\ x_7 \ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ 0 \end{array} = \begin{array}{c} 9 \\ 8 \ 7 \\ 6 \ 5 \ 6 \\ 4 \ 5 \ 4 \\ 3 \ 2 \\ 1 \\ 0 \end{array}, \quad \begin{array}{c} 9 \\ 8 \ 7 \\ 7 \ 5 \ 6 \\ 4 \ 5 \ 4 \\ 3 \ 2 \\ 1 \\ 0 \end{array} \quad \text{and} \quad \begin{array}{c} 9 \\ 8 \ 7 \\ 7 \ 5 \ 6 \\ 6 \ 4 \ 5 \ 4 \\ 3 \ 2 \\ 1 \\ 0 \end{array}$$





From the table, we obtain $h^{2,1}(Y) = 37$, hence $\chi(Y) = 2(h^{1,1}(Y) - h^{2,1}(Y)) = -64$.

Let us recall that the conifold transition is a surgery of Calabi–Yau 3-folds replacing finite vanishing S^3 by the same number of exceptional $\mathbb{P}^1 \simeq S^2$ as in [Cle]. From the inclusion-exclusion principle of the Euler numbers, $\chi(X)$ and $\chi(Y)$ are related with each other as:

$$\chi(X) = \chi(Y) - 2p,$$

where p is the number of nodes on X_0 . Then we need to know the total degree of codimension three singular loci of $\mathbb{P}_{\Delta(P)}$, which equals the number $p / \prod_j d_j = p$ in our case. From Proposition 2.18 (2), an irreducible component of singular loci of the Hibi toric variety $\mathbb{P}_{\Delta(P)}$ corresponds to a minimal convex cycle in \hat{P} . There are four such cycles (or boxes) b_1, \dots, b_4

and all of them define the codimension three faces in $\Delta(P)$ as in Proposition 2.16. Again we can compute the corresponding index poset P' of them by the method used above. The resulting posets P' are summarized as follows.

| singular loci | b_1 | b_2 | b_3 | b_4 |
|---------------|---|---|---|--|
| |  |  |  |  |
| degree | 5 | 3 | 2 | 9 |

From Proposition 2.18 (3), we can compute the degree of each irreducible component of singular loci by counting the maximal chains in $J(P')$. Then we obtain that total degree p , that is, the number of nodes on X_0 is 19. We conclude $\chi(X) = -102$. \square

Remark 4.2. (1) The existence of the Calabi–Yau 3-fold with these topological invariants were previously conjectured by [vEvS] from the monodromy calculations of Calabi–Yau differential equations. We also perform the similar calculation in the next section.

(2) It may be possible to calculate the Euler number $\chi(X)$ in another way, by computing the Chern–Mather class of the Schubert variety Σ . For the Grassmann Schubert varieties, this is done by [Jon] using Zelevinsky’s IH -small resolution. In our case, however, it is known that Σ does not admit any IH -small resolution [Per1].

5. MIRROR SYMMETRY

In this section, we study the mirror symmetry for Calabi–Yau complete intersections in Gorenstein Hibi toric varieties and those in minuscule Schubert varieties. In Subsection 5.2, we focus on the specific example $X = \Sigma(1^9)$ and carry out further calculations.

5.1. Mirror constructions. Our study of the family of affine complete intersections in $(\mathbb{C}^*)^n$ which are expected to be birational to the conjectural mirror family for Calabi–Yau complete intersections in minuscule Schubert varieties is based on the general method via conifold transition proposed by [BCFKvS1] [Bat2]. First we apply the Batyrev–Borisov mirror construction [Bat1] [Bor] to Calabi–Yau complete intersections in Gorenstein Hibi toric varieties. Second we explain the conjectural mirror construction via conifold transition and describe a family of affine complete intersections in $(\mathbb{C}^*)^n$ conjecturally birational to the mirror family for Calabi–Yau complete intersections in minuscule Schubert varieties. Finally we establish the closed formula for the fundamental period.

5.1.1. Batyrev–Borisov construction. Let P be a finite pure poset and $J(P)$ the associated distributive lattice of order ideals. Let $N = \mathbb{Z}P$ and $M = \text{Hom}(N, \mathbb{Z})$ be dual free abelian groups of rank $|P|$, and $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ be the real scalar extension of N and M , respectively. Recall that the order polytope $\Delta(P) \subset M_{\mathbb{R}}$ is a $|P|$ -dimensional integral polytope associated with the hyperplane class on the Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$. By Proposition 2.18 and Theorem 4.1.9 in [Bat1], a unimodular transformation of $h_P \Delta(P)$ becomes a *reflexive polytope* corresponding to the anticanonical sheaf $-\mathcal{K}_{\mathbb{P}_{\Delta(P)}} = \mathcal{O}(h_P)$, i.e., it contains the unique internal

integral point 0 and every facet has integral distance one to 0. For instance, we can choose $\Delta := \sum_{u \in P} h(u)\chi_u - h_P\Delta(P)$ as such a reflexive polytope.

The polar dual polytope Δ^* of Δ also has a good description [BCFKvS2] [HH]. The set of edges E in the Hasse diagram of \hat{P} plays an important role. We remark that the abelian groups $\mathbb{Z}\hat{P} = N \oplus \mathbb{Z}\{\hat{0}, \hat{1}\}$ and $\mathbb{Z}E$ can be viewed as the groups of 0-chains and 1-chains of the natural chain complex associated with the Hasse diagram of \hat{P} . The boundary map in the chain complex is

$$\partial : \mathbb{Z}E \longrightarrow \mathbb{Z}\hat{P}, \quad e \mapsto t(e) - s(e).$$

We also consider the projection $\text{pr}_1 : \mathbb{Z}\hat{P} \rightarrow N$ and the composed map

$$\delta := \text{pr}_1 \circ \partial : \mathbb{Z}E \longrightarrow N.$$

In this notation, the dual polytope Δ^* coincides with the convex hull of the image $\delta(E) \subset N_{\mathbb{R}}$. Further, the linear map δ gives a bijection between E and the set of vertices in Δ^* .

Let D_e denote the toric divisor on $\mathbb{P}_{\Delta(P)}$ associated with an edge $e \in E$ under the above bijection and $D_{E'} := \sum_{e \in E'} D_e$ for any subset $E' \subset E$. It is easy to see that there is a linear equivalence relation

$$\sum_{s(e)=u} D_e \simeq \sum_{t(e)=u} D_e,$$

for each $u \in P$, and that $\mathcal{O}(D_{E^k})$ coincides with a very ample invertible sheaf $\mathcal{O}(1)$ associated with $\Delta(P)$ for each $E^k := \{e \in E \mid h(s(e)) = k\}$. Let $X_0 \subset \mathbb{P}_{\Delta(P)}$ be a general Calabi–Yau complete intersection of degree (d_1, \dots, d_r) with respect to $\mathcal{O}(1)$. That is, d_1, \dots, d_r satisfies $\sum_{j=1}^r d_j = h_P$. We choose a *nef-partition* of Δ , a special kind of Minkowski sum decomposition $\Delta = \Delta_1 + \dots + \Delta_r$ of Δ , in the following specific way. Define subsets E_j of edges in E as

$$E_j = \bigcup_{k=d_1+\dots+d_{j-1}+1}^{d_1+\dots+d_j} E^k.$$

It turns out that $\mathcal{O}(D_{E_j}) = \mathcal{O}(d_j)$ and the nef-partition is arisen from $E = E_1 \cup E_2 \cup \dots \cup E_r$. Define $\nabla_j = \text{Conv}(\{0\}, \delta(E_j))$ and the Minkowski sum $\nabla = \nabla_1 + \dots + \nabla_r \subset N_{\mathbb{R}}$. From [Bor], it holds that

$$\Delta^* = \text{Conv}(\nabla_1, \dots, \nabla_r), \quad \nabla^* = \text{Conv}(\Delta_1, \dots, \Delta_r) \quad \text{and} \quad \Delta = \Delta_1 + \dots + \Delta_r,$$

where Δ_j is the integral polytope in $M_{\mathbb{R}}$ defined by $\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$. One may be also able to write down more precise combinatorial expressions for Δ_j and ∇^* .

Now we introduce the Batyrev–Borisov mirror of $Y = \hat{X}_0$, the strict transform of X_0 in a MPCP-resolution $\hat{\mathbb{P}}_{\Delta(P)}$ of $\mathbb{P}_{\Delta(P)}$. The mirror of $Y \subset \hat{\mathbb{P}}_{\Delta(P)}$ is birational to the set given by the following equations in torus $(\mathbb{C}^*)^{|P|}$:

$$\tilde{f}_j = 1 - \left(\sum_{e \in E_j} a_e t^{\delta(e)} \right) = 0 \quad (\text{for all } 1 \leq j \leq r),$$

where each $a_e \in \mathbb{C}$ is a parameter. Further, the smooth mirror manifold Y^* of Y is also obtained as the closure of the above set in MPCP-resolution $\hat{\mathbb{P}}_{\nabla}$ of \mathbb{P}_{∇} .

In general, the mirror manifold $Y^* \subset \hat{\mathbb{P}}_{\nabla}$ actually has the expected stringy (or string-theoretic) Hodge numbers as proved in [BB1, BB2]. The stringy Hodge numbers of X_0 coincide

with the usual Hodge numbers of Y if there exists a crepant resolution $Y \rightarrow X_0$. Applying their formula for stringy $(1, *)$ -Hodge numbers to the case of Calabi–Yau complete intersections X_0 in Gorenstein Hibi toric varieties $\mathbb{P}_{\Delta(P)}$, we obtain the following convenient expressions in terms of the poset P , which we already used in Proof of Proposition 4.1.

Theorem 5.1 (cf. [BB1, Proposition 8.6]). *The stringy $(1, *)$ -Hodge numbers of a general Calabi–Yau complete intersections X_0 of degree (d_1, \dots, d_r) in a Gorenstein Hibi toric variety $\mathbb{P}_{\Delta(P)}$ are given by the following formulae*

$$h_{\text{st}}^{1,1}(X_0) = |E| - |P|, \quad h_{\text{st}}^{1,k}(X_0) = 0 \quad (1 < k < |P| - r - 1),$$

$$h_{\text{st}}^{1,|P|-r-1}(X_0) = \sum_{i \in I} \left[\sum_{J \subset I} (-1)^{|J|} l((d_i - d_J)\Delta(P)) \right] - \sum_{J \subset I} (-1)^{r-|J|} \left[\sum_{e \in E} l^*(d_J \theta_e) \right] - |P|,$$

where $I = \{1, \dots, r\}$, $d_J := \sum_{j \in J} d_j$ and θ_e is the facet of P corresponding to the edge $e \in E$. The nonzero contribution in the first term of $h_{\text{st}}^{1,|P|-r-1}(X_0)$ comes only from the range of $d_i - d_J \geq 0$ and in the second term from that of $d_J = h_P - 1$ or h_P .

5.1.2. Construction via conifold transition. A conifold transition of a smooth Calabi–Yau 3-fold X is the composite operation of a flat degeneration of X to X_0 with finitely many nodes and a small resolution $Y \rightarrow X_0$. The conjecture proposed in [BCFKvS1] is that the mirror Calabi–Yau 3-folds Y^* and X^* are again related in the same way. The construction is depicted as the following diagram, Figure 4. In the diagram, dashed and solid arrows represent flat degenerations and small contraction morphisms respectively.

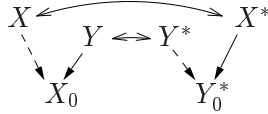


FIGURE 4. Mirror symmetry and conifold transitions

By an argument in [Bat2] on generalized monomial-divisor correspondence, there is a natural specialization Y_0^* of the family of Y^* to get the mirror of X . That is, the specialized parameter $(a_e)_{e \in E}$ should be $\Sigma(\Delta^*)$ -admissible, i.e. there exists a $\Sigma(\Delta^*)$ -piecewise linear function $\phi : N_{\mathbb{R}} \rightarrow \mathbb{R}$ corresponding to a Cartier divisor on X such that $\phi \circ \delta(e) = \log |a_e|$. In all our case, $\text{Pic } \mathbb{P}_{\Delta(P)} \simeq \text{Pic } X \simeq \mathbb{Z}$ holds. Then we can simply specialize the family to be diagonal, i.e. setting all the coefficients a_e to be a same parameter a . Now we repeat the conjecture of [BCFKvS1] as a specific form in our terminology combining with the argument on the number of nodes in Proof of Proposition 4.1.

Conjecture 5.2 (cf. [BCFKvS1, Conjecture 6.1.2]). *Let p be a number of nodes on a Calabi–Yau 3-fold $X_0 \subset \mathbb{P}_{\Delta(P)}$, namely, $p = (\prod_{j=1}^r d_j) \sum c_{|P'|}$ where the sum is over all minimal convex cycles in P corresponding to the contractions $P \rightarrow P'$ with $|P'| = |P| - 3$ and $c_{|P'|}$*

denotes the number of maximal chains in $J(P')$. We define a one parameter family of affine complete intersections in $(\mathbb{C}^*)^{|P|}$ defined by the following equations:

$$f_j = 1 - a \left(\sum_{e \in E_j} t^{\delta(e)} \right) = 0 \quad (\text{for all } 1 \leq j \leq r).$$

The closure Y_0^* of the above set in a MPCP-resolution $\hat{\mathbb{P}}_\nabla$ has p nodes, satisfying $(|E| - |P| - p - 1)$ relations for general a . A small resolution $X^* \rightarrow Y_0^*$ is a mirror manifold of X with the correct Hodge numbers, $h^{i,j}(X^*) = h^{3-j,i}(X)$.

Remark 5.3. In the case of smoothing of 3-dimensional Calabi–Yau hypersurfaces in Gorenstein Hibi toric varieties, we can see Conjecture 5.2 holds by the same argument as in [BCFKvS1] [BK], i.e., the polar duality of faces gives a one-to-one correspondence between singular $\mathbb{P}^1 \subset \mathbb{P}_{\Delta(P)}$ and torus invariant $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}_{\Delta^*}$ which intersect non-transversally with the closure of the set $\{f_1 = 0\}$, and the MPCP-resolution $\hat{\mathbb{P}}_{\Delta^*} \rightarrow \mathbb{P}_{\Delta^*}$ increases them by h_P times.

In general, we can define a one parameter family Y_0^* for any dimensional Calabi–Yau variety $X = X(w)(d_1, \dots, d_r)$. In the remaining part, we refer to not only X^* but also Y_0^* as a *conjectural mirror* of X .

5.1.3. Fundamental period. We derive the explicit form of the fundamental period for the conjectural mirror family of a Calabi–Yau variety $X(w)(d_1, \dots, d_r)$. Obviously, the coordinate transformation $t_u \rightarrow \zeta^{h_u} t_u$ gives a \mathbb{Z}_{h_P} -symmetry $a \rightarrow \zeta a$ in this family, where $\zeta = e^{2\pi\sqrt{-1}/h_P}$. Therefore we should take $x := a^{h_P}$ as a genuine moduli parameter. The fundamental period $\omega_0(x)$ of the mirror family is defined by integration of the holomorphic $(|P| - r)$ -form Ω_x on a (real) torus cycle \mathbb{T} that vanishes at $x = 0$. By residue theorem, we obtain the following formula up to the constant multiplication,

$$\begin{aligned} \omega_0(x) &= \int_{\mathbb{T}} \Omega_x = \frac{1}{(2\pi\sqrt{-1})^{|P|}} \int_{|t_u|=1} \frac{1}{\prod_{j=1}^r f_j} \prod_{i=1}^{|P|} \frac{dt_i}{t_i} \\ &= \sum_{m=0}^{\infty} a^{h_P m} \frac{1}{(2\pi\sqrt{-1})^{|P|}} \int_{|t_u|=1} \prod_{j=1}^r \left(\sum_{e \in E_j} t^{\delta(e)} \right)^{d_j m} \prod_{i=1}^{|P|} \frac{dt_i}{t_i} \\ &= \sum_{m=0}^{\infty} x^m N(J_j, E_j) = \sum_{m=0}^{\infty} x^m \frac{\prod_{j=1}^r (d_j m)!}{m!^{h_P}} N(J^k, E^k), \end{aligned}$$

where $J_j(m) := \{(j, i) \mid 1 \leq i \leq d_j m\}$, $J^k(m) := \{(k, i) \mid 1 \leq i \leq m\} \subset \mathbb{N}^2$ and $N(J_j, E_j) := \# \left\{ \phi: \bigcup_j J_j(m) \rightarrow E \mid \phi(J_j(m)) \subset E_j, \sum_{s(e)=u} \phi^{-1}(e) = \sum_{t(e)=u} \phi^{-1}(e) \right\}$.

In the case that the Hasse diagram of P (and hence \hat{P}) is a plane graph, we can go further like [BCFKvS2]. This is originally formulated in the work of Bondal and Galkin [BG] for the Landau–Ginzburg mirror of minuscule G/Q . For the mirror of complete intersections in general minuscule Schubert varieties, we can also follow the same road because the Hasse diagram of a minuscule poset is always a plane graph. If the Hasse diagram of P is a plane graph, we can define the dual graph B of the Hasse diagram of \hat{P} on a sphere $S^2 = \mathbb{P}^1$ with putting $\hat{1}, \hat{0}$ on $\pm\sqrt{-1}\infty$ respectively. We denote by b_L, b_R the elements $b \in B$ corresponding

to the farthest left and right areas respectively. We draw the Hasse diagram of \hat{P} and its dual graph B below, Figure 5, for the minuscule poset of $G(2, 6)$ as an example.

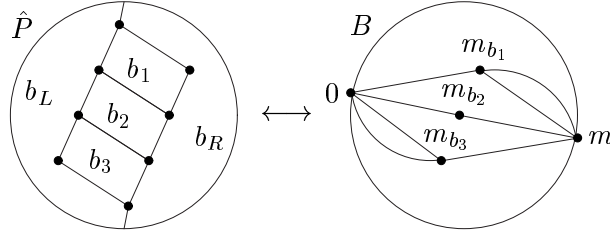


FIGURE 5. An example of \hat{P} and the dual graph B

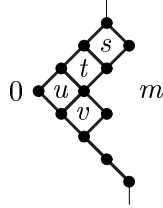
The orientation of an edge e of B is defined as the direction from the left $l(e)$ to the right $r(e)$. We attain the variable m_b for each element $b \in B$ and set $m_{b_L} = 0$ and $m_{b_R} = m$.

Proposition 5.4. *Let $X(w)$ be a minuscule Schubert variety and $P = P_w$ its minuscule poset. Under the above notation, the fundamental period $\omega_0(x)$ for the conjectural mirror family of a Calabi–Yau complete intersection $X(w)(d_1, \dots, d_r)$ is presented in the following:*

$$\omega_0(x) = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^r (d_j m)!}{m!^{h_P}} \sum_{m_b \in B} \prod_{e \in E(B)} \binom{m_{r(e)}}{m_{l(e)}} x^m.$$

5.2. Mirror symmetry for $\Sigma(1^9)$.

5.2.1. *Picard–Fuchs equation.* Now we focus on the Calabi–Yau 3-fold $X = \Sigma(1^9)$. The fundamental period of the mirror family can be read from the following diagram.



The vertices of the dual graph B corresponds to the separated areas. The fundamental period turns out to be

$$\omega_0(x) = \sum_{m,s,t,u,v} \binom{m}{s}^2 \binom{m}{v}^2 \binom{m}{t} \binom{s}{t} \binom{t}{u} \binom{v}{u} x^m,$$

where $x = a^9$. With the aid of numerical method, we obtain the Picard–Fuchs equation for the conjectural mirror of X .

Proposition 5.5. *Let $\omega_0(x)$ be the above power series around $x = 0$, which corresponds to the fundamental period for the conjectural mirror family of the Calabi–Yau 3-fold $X = \Sigma(1^9)$.*

This satisfies the Picard–Fuchs equation $\mathcal{D}_x \omega_0(x) = 0$ with $\theta_x = x \partial_x$ and

$$\begin{aligned} \mathcal{D}_x = & 121\theta_x^4 - 77x(130\theta_x^4 + 266\theta_x^3 + 210\theta_x^2 + 77\theta_x + 11) \\ & - x^2(32126\theta_x^4 + 89990\theta_x^3 + 103725\theta_x^2 + 55253\theta_x + 11198) \\ & - x^3(28723\theta_x^4 + 74184\theta_x^3 + 63474\theta_x^2 + 20625\theta_x + 1716) \\ & - 7x^4(1135\theta_x^4 + 2336\theta_x^3 + 1881\theta_x^2 + 713\theta_x + 110) - 49x^5(\theta_x + 1)^4. \end{aligned}$$

5.2.2. *Monodromy calculation.* The differential operator \mathcal{D}_x in Proposition 5.5 is already appeared in the lists of [AvEvSZ, vEvSdb] with conjectural topological invariants, obtained from the monodromy calculation for a particular basis of the solutions for \mathcal{D}_x and some assumptions by mirror symmetry [vEvS]. The existence of the smooth Calabi–Yau 3-fold $X = \Sigma(1^9)$ and our computation of topological invariants in Proposition 4.1 answer to their conjecture. On the other hand, we observe that there exists another MUM-point for this equation, not only $x = 0$ but $x = \infty$. In fact, the Riemann scheme of the differential operator \mathcal{D}_x is

$$P \left\{ \begin{array}{c|ccccc} \zeta_1 & -11/7 & \zeta_2 & 0 & \zeta_3 & \infty \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 3 & 1 & 0 & 1 & 1 \\ 2 & 4 & 2 & 0 & 2 & 1 \end{array} \right\},$$

where $\zeta_1 < \zeta_2 < \zeta_3$ are the roots of the discriminant $x^3 + 159x^2 + 84x - 1$. The singularities at $x = \zeta_1, \zeta_2, \zeta_3$ are called conifold and there is no monodromy around the point $x = -11/7$, called apparent singularity. We believe that the point $x = \infty$ has also geometric interpretation.

Recall the argument in [CdOGP] based on the mirror symmetry, which involves an integral symplectic basis of solutions in the original Calabi–Yau geometry. Let us start from the Frobenius basis of solutions for \mathcal{D}_x , namely the unique normalized regular power series solution $\omega_0(x) = 1 + O(x)$ and the followings:

$$\begin{aligned} \omega_1(x) &= \omega_0(x) \log x + \omega_1^{\text{reg}}(x), \\ \omega_2(x) &= \omega_0(x) (\log x)^2 + 2\omega_1^{\text{reg}}(x) \log x + \omega_2^{\text{reg}}(x), \\ \omega_3(x) &= \omega_0(x) (\log x)^3 + 3\omega_1^{\text{reg}}(x) (\log x)^2 + 3\omega_2^{\text{reg}}(x) \log x + \omega_3^{\text{reg}}(x), \end{aligned}$$

where $\omega_k^{\text{reg}}(x)$ is a regular power series around $x = 0$ without constant term. We expect an integral symplectic basis has the following form:

$$\Pi^X(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta/24 & a & -\kappa/2 & 0 \\ \gamma & \beta/24 & 0 & \kappa/6 \end{pmatrix} \begin{pmatrix} n_0 \omega_0(x) \\ n_1 \omega_1(x) \\ n_2 \omega_2(x) \\ n_3 \omega_3(x) \end{pmatrix},$$

where $\kappa = -\deg(X)$, $\beta = -c_2(X) \cdot H$, $\gamma = -n_3 \zeta(3) \chi(X)$, $n_k = 1/(2\pi i)^k$ with the topological invariants of X in Proposition 4.1 and a is an unknown parameter without geometric interpretation although it may be consistent to choose $a \in \deg(X)/2 + \mathbb{Z}$.

If there exists a Calabi–Yau geometry around $x = \infty$, we also expect the existence of similar basis $z\Pi^Z(z)$ of solutions for \mathcal{D}_z under some appropriate coordinate change $z = c/x$.

We denote by $\Pi^Z(z)$ the gauge fixed basis, exactly the same form as $\Pi^X(x)$ with the Frobenius basis $\omega_k^Z(z)$ for $\mathcal{D}_z^Z = z\mathcal{D}_z z^{-1}$ and unknown parameters $\deg(Z)$, $c_2(Z) \cdot H$, $\chi(Z)$ and a^Z .

Once passing to a numerical calculation like [vEvS], we obtain the following results.

- (1) There exists the integral symplectic basis $\Pi^X(x)$ and $z\Pi^Z(z)$ with the parameters,

$$a = a^Z = -1/2, \quad c = -1,$$

$$\deg(Z) = 21, \quad c_2(Z) \cdot H = 66, \quad \chi(Z) = -102.$$

- (2) The analytic continuation along a path in the upper half plane gives the following relation of two basis $\Pi^X(x)$ and $z\Pi^Z(z)$,

$$\Pi^X(x) = N_z S_{xz} z\Pi^Z(z),$$

with $N_z = 1$ and the symplectic matrix $S_{xz} = \begin{pmatrix} 8 & 4 & 2 & 5 \\ 4 & 0 & 1 & 2 \\ 10 & -25 & 2 & -1 \\ -21 & 2 & -5 & -10 \end{pmatrix}$.

- (3) With respect to the local basis of the analytic continuation of $\Pi^X(x)$ and $z\Pi^Z(z)$ along a path in the upper half plane, the monodromy matrices M_p^X and M_p^Z at each singular point $x = -\frac{1}{z} = \zeta_1, \zeta_2, 0, \zeta_3, \infty$ have the following form, respectively:

| | | | | | |
|---------|--|--|---|--|---|
| M_p^X | $\begin{pmatrix} 169 & -80 & 32 & 64 \\ 84 & -39 & 16 & 32 \\ 210 & -100 & 41 & 80 \\ -441 & 210 & -84 & -167 \end{pmatrix}$ | $\begin{pmatrix} 13 & -8 & 2 & 4 \\ 6 & -3 & 1 & 2 \\ 24 & -16 & 5 & 8 \\ -36 & 24 & -6 & -11 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 16 & 33 & 1 & 0 \\ -12 & -17 & -1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 286 & -130 & 55 & 111 \\ 89 & -43 & 17 & 34 \\ -307 & 127 & -60 & -122 \\ -465 & 218 & -89 & -179 \end{pmatrix}$ |
| M_p^Z | $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -9 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 343 & -17 & 83 & 168 \\ 104 & -9 & 25 & 50 \\ -496 & 8 & -121 & -247 \\ -432 & 32 & -104 & -209 \end{pmatrix}$ | $\begin{pmatrix} 211 & -20 & 50 & 100 \\ 105 & -9 & 25 & 50 \\ 42 & -4 & 11 & 20 \\ -441 & 42 & -105 & -209 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 10 & 21 & 1 & 0 \\ -9 & -11 & -1 & 1 \end{pmatrix}$ |

TABLE 2. Monodromy matrices

All in the above results strongly indicate the existence of the geometric interpretation at $x = \infty$. Thus we are led to the following conjecture based on the homological mirror symmetry similar to the examples of the Grassmannian–Pfaffian in [Rød] and the Reye congruence Calabi–Yau 3-fold in [HT1].

Conjecture 5.6. *There exists a smooth Calabi–Yau 3-fold Z with $h^{1,1} = 1$ whose derived category of coherent sheaves is equivalent to that of the Calabi–Yau 3-fold $X = \Sigma(1^9)$. The topological invariants of Z are*

$$\deg(Z) = 21, \quad c_2(Z) \cdot H = 66, \quad \chi(Z) = -102,$$

where H is the ample generator of the Picard group $\text{Pic}(Z) \simeq \mathbb{Z}$.

The Calabi–Yau 3-fold Z in Conjecture 5.6 can not be birational to X because $h^{1,1} = 1$ and $\deg(Z) \neq \deg(X)$, so that it should be a non-trivial Fourier–Mukai partner of X .

5.2.3. BPS numbers. As a further consistency check in Conjecture 5.6 and an application of the mirror construction, we carry out the computation of BPS numbers by using the methods proposed by [CdOGP] [BCOV1, BCOV2]. The BPS numbers $n_g(d)$ are related with the Gromov–Witten invariants $N_g(d)$ by some closed formula [GV], hence we obtain the prediction for Gromov–Witten invariants of X and Z from these computations.

We skip all the details here and only present the results in Appendix B. For the details, one can get many references in now. Here we have followed [HK], where a very similar example to ours, the Grassmannian–Pfaffian Calabi–Yau 3-fold, has been analyzed.

APPENDIX A. OTHER EXAMPLES IN MINUSCULE G/Q

We remark that there are some examples of Calabi–Yau complete intersections in minuscule homogeneous spaces G/Q that get attention. In these cases one may be able to know more details because the structure of the small quantum cohomology ring $QH^*(G/Q)$ of G/Q has been revealed intensively by recent researches [Ber] [KT2] [CMP] for instance.

In particular, the quantum Chevalley formula [CMP, Proposition 4.1] for minuscule homogeneous spaces G/Q is given in terms of the minuscule poset P_w as follows

$$[H] * [X(w)] = \sum_{w \text{ covers } w'} [X(w')] + q[X(w)]^{\text{PD}},$$

where q is the quantum parameter and $[X(w)]^{\text{PD}}$ denotes the Poincaré dual of $[X(w)]$, whose minuscule poset corresponds to the order dual of $P_Q \setminus P_w$. Of course the limit $q \rightarrow 0$ coincides with the classical Chevalley formula in Remark 2.6. From this formula, we can calculate the quantum differential system of minuscule G/Q and reduce this first order system to a higher order differential system by some computer aided calculation (see [BCFKvS1] [Gue] for instance). For example, the orthogonal Grassmannian $OG(5, 10)$ and the Cayley plane \mathbb{CP}^2 have the quantum differential operator annihilating the component associated with the fundamental class as follows:

$$\begin{aligned} & \theta_q^{11}(\theta_q - 1)^5 - q\theta_q^5(2\theta_q + 1)(17\theta_q^2 + 17\theta_q + 5) + q^2, \\ & \theta_q^{17}(\theta_q - 1)^9 - 3q\theta_q^9(2\theta_q + 1)(3\theta_q^2 + 3\theta_q + 1)(15\theta_q^2 + 15\theta_q + 4) - 3q^2(3\theta_q + 2)(3\theta_q + 4), \end{aligned}$$

respectively, where $\theta_q = q\partial_q$. By the quantum hyperplane section theorem for homogeneous spaces [Kim], we can determine the genus zero Gromov–Witten potential for each example of Calabi–Yau complete intersections in minuscule G/Q . In fact this procedure also works for general rational homogeneous spaces G/Q by replacing the above quantum Chevalley formula by the general one obtained in [FW] though we do not go into the specifics here.

On the other hand, we already know the conjectural mirror family. Hence comparing both sides, we can prove the classical genus zero ‘mirror theorem’ [Giv] [LLY] for these examples combining Conjecture 5.2. In particular, we can do this for $OG(5, 10)(1^6, 2)$, the remained example of smooth complete intersection Calabi–Yau 3-folds in minuscule Schubert varieties in Proposition 3.1. The Picard–Fuchs operator for the conjectural mirror family of $OG(5, 10)(1^6, 2)$ coincides with the conjecture in [vEvS],

$$\theta_x^4 - 2x(2\theta_x + 1)^2(17\theta_x^2 + 17\theta_x + 5) + 4x^2(\theta_x + 1)^2(2\theta_x + 1)(2\theta_x + 3),$$

where $\theta_x = x\partial_x$.

Other dimensional examples also seem meaningful. The $K3$ surface $OG(5, 10)(1^8)$ is an example which has a nontrivial Fourier–Mukai partner [Muk], whose mirror symmetry are investigated in [HLOY]. The Picard–Fuchs operator for the mirror family of $OG(5, 10)(1^8)$ is

$$\theta_x^3 - x(2\theta_x + 1)(17\theta_x^2 + 17\theta_x + 5) + x^2(\theta_x + 1)^3,$$

where $\theta_x = x\partial_x$. It has exactly the same property as the Picard–Fuchs operators of conjectural mirror families for $G(2, 7)(1^7)$ and $\Sigma(1^9)$, with two MUM-points. Further this operator is also well-known since it is used essentially in Apéry’s proof of irrationality of $\zeta(3)$, with that of the mirror family for $G(2, 5)(1^5)$ in $\zeta(2)$ case. From this viewpoint, it may be valuable that the conjectural mirror family of Calabi–Yau 4-fold $\mathbb{O}\mathbb{P}^2(1^{12})$ also has the following 5th-order Picard–Fuchs operator,

$$\theta_x^5 - 3x(2\theta_x + 1)(3\theta_x^2 + 3\theta_x + 1)(15\theta_x^2 + 15\theta_x + 4) - 3x^2(\theta_x + 1)^3(3\theta_x + 2)(3\theta_x + 4),$$

where $\theta_x = x\partial_x$. This operator is related to the value $\zeta(4)$ [Zud] [AZ]. The relationship between the quantum differential system of rational homogeneous spaces and zeta values are also numerically confirmed in [Gal] with the definition of the Apéry characteristic class.

APPENDIX B. BPS NUMBERS

| d | $g = 0$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ |
|-----|-------------------|------------------|----------------|--------------|----------|
| 1 | 252 | 0 | 0 | 0 | 0 |
| 2 | 1854 | 0 | 0 | 0 | 0 |
| 3 | 27156 | 0 | 0 | 0 | 0 |
| 4 | 567063 | 0 | 0 | 0 | 0 |
| 5 | 14514039 | 4671 | 0 | 0 | 0 |
| 6 | 424256409 | 1029484 | 0 | 0 | 0 |
| 7 | 13599543618 | 112256550 | 5058 | 0 | 0 |
| 8 | 466563312360 | 9161698059 | 7759089 | 0 | 0 |
| 9 | 16861067232735 | 645270182913 | 2496748119 | 151479 | 0 |
| 10 | 634912711612848 | 41731465395267 | 438543955881 | 418482990 | -3708 |
| 11 | 24717672325914858 | 2557583730349461 | 56118708041940 | 217285861284 | 33975180 |

TABLE 3. BPS numbers $n_g^X(d)$ of X

| d | $g = 0$ | $g = 1$ | $g = 2$ | $g = 3$ | $g = 4$ |
|-----|--------------------|-------------------|------------------|---------------|-------------|
| 1 | 387 | 0 | 0 | 0 | 0 |
| 2 | 4671 | 0 | 0 | 0 | 0 |
| 3 | 124323 | 1 | 0 | 0 | 0 |
| 4 | 4782996 | 1854 | 0 | 0 | 0 |
| 5 | 226411803 | 606294 | 0 | 0 | 0 |
| 6 | 12249769449 | 117751416 | 27156 | 0 | 0 |
| 7 | 727224033330 | 17516315259 | 33487812 | 252 | 0 |
| 8 | 46217599569117 | 2252199216735 | 15885697536 | 7759089 | 0 |
| 9 | 3094575464496057 | 265984028638047 | 4690774243470 | 13680891072 | 1127008 |
| 10 | 215917815744645750 | 29788858876065588 | 1053460470463461 | 9429360817149 | 12259161360 |

TABLE 4. BPS numbers $n_g^Z(d)$ of Z

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